Image Processing Fundamentals

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Do What? How?

- Find the surface normal to the embedded surface?
- Locate an object within the volume?
- Take the derivative of a kidney?

- How do I evaluate the error introduced by sampling?
- What is aliasing, really?
- How does this all work together in volume graphics?

Image Basics

- Given a discrete volume dataset, *vol*[x][y][z]
- Imagine a volume generating function, f, such that for any particular point $p_i = (x_i, y_i, z_i)$:

  $f(x_i, y_i, z_i) = vol[x_i][y_i][z_i]$

- Must reconstruct f(x,y,z) from *vol*[x][y][z].
- Can make measurements of f(x,y,z), (e.g.,derivatives)
- HOW?

Convolution

- Sampling.
- Interpolation.
- Reconstruction.
- Low pass filtering.
- Noise suppression.
- Edge enhancement.
- Derivative measurement.
- Linear scale space analysis.

Convolution

Convolution integral

$$\boxed{h(x) \otimes I(x) = \int_{-\infty}^{\infty} h(\tau) I(x-\tau)\, d\tau}$$

Convolution & the Gaussian

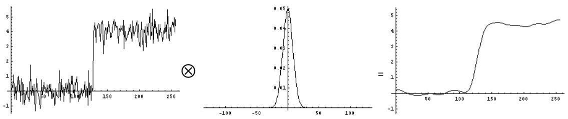
Convolution integral

$$\boxed{h(x) \otimes I(x) = \int_{-\infty}^{\infty} h(\tau) I(x-\tau)\, d\tau}$$

Gaussian as a convolution kernel (with spatial scale parameter, σ)

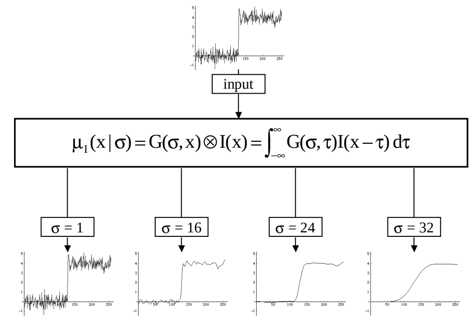
$$\boxed{h(x) = G(\sigma, x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}}$$

Smoothing - noise filtering

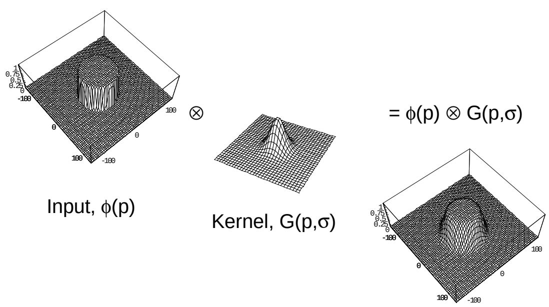


Input, $\phi(x)$ ⊗ Kernel, $G(x,\sigma)$ $= \phi(x) \otimes G(x,\sigma)$

Linear Scale Space



Convolution in 2D



Input, $\phi(p)$ ⊗ Kernel, $G(p,\sigma)$ $= \phi(p) \otimes G(p,\sigma)$

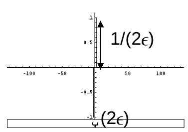
Properties of the Convolution Operator

Property:	Mathematically:
• Commutative.	• $f \otimes h = h \otimes f$
• Associative	• $(f \otimes h) \otimes g = f \otimes (h \otimes g)$
• Distributive over addition.	• $f \otimes (h + g) = (f \otimes h) + (f \otimes g)$

Convolution & differentiation

- Problem: Real-world data are seldom accompanied with continuous functions, suitable for differentiation.
- Observation: Differentiation is just a convolution!

$$\frac{\partial}{\partial x}\phi = \frac{\partial}{\partial x} \otimes \phi$$

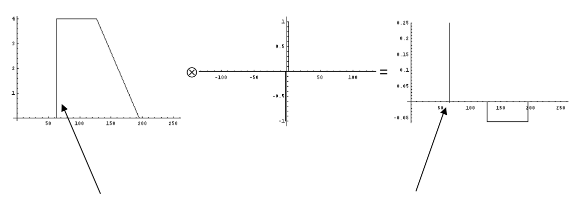


- Problem: Derivatives of real-world data are not always defined.
- Observation: Differentiation is associative and commutative.

$h(x) \otimes \partial/\partial x \otimes f(x) = \partial/\partial x \otimes h(x) \otimes f(x)$

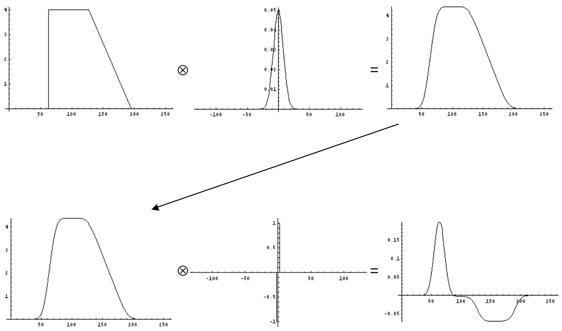
- Solution: Combine differentiation with a regularizing kernel.

Differentiation

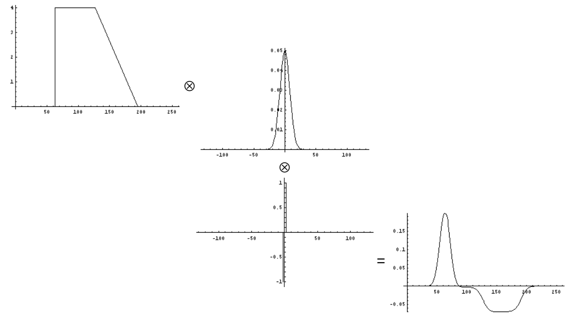


Not-differentiable NaN

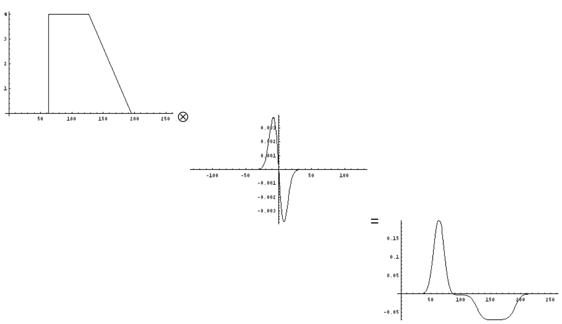
Differentiation



Differentiation



Differentiation



Now for something different…

But not completely different…

$e^{i\pi} = -1$

$e^{\alpha+i\beta} = \cos \alpha + i \sin \beta$

The Fourier Transform

- A different mirror with which to view images.
- The Fourier transform

$$\mathcal{F}(\phi(x)) = \int_{-\infty}^{\infty} \phi(x) e^{-i2\pi x\nu} dx = \Phi(\nu)$$

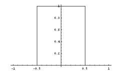
- The result transforms a function of x in image space to a function of ν in frequency space.
- There is an Inverse Fourier transform for undoing this process.

$$\mathcal{F}^{-1}(\Phi(\nu)) = \int_{-\infty}^{\infty} \Phi(\nu) e^{i2\pi x\nu} d\nu = \phi(x)$$

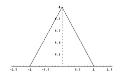
Fourier Transforms of common functions

Image Space:

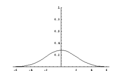
Box
(nearest neighbor)



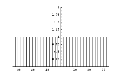
Pyramid
(linear interpolant)



Gaussian
(std. distribution)

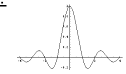


Comb / Shah
(sampling function)

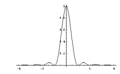


Frequency Space:

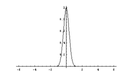
F(Box)
sinc



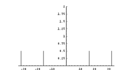
F(Pyramid)
(linear interpolant)



Gaussian
(non-normalized)



Comb



Properties of the Fourier Transform

Property:

- Frequency Scaling $\Longrightarrow$
  Inverse Spatial Scale Change

$$\mathcal{F}\frac{1}{|a|}\phi\left(\frac{\nu}{a}\right)(\phi(ax)) = \Phi(a\nu)$$

and

$$\mathcal{F}^{-1}\frac{1}{|a|}\Phi\left(\frac{\nu}{a}\right) = \phi(ax)$$

Property:

- Spatial Scaling $\Longrightarrow$
  Inverse Frequency Scale Change

$$\mathcal{F}(\phi(ax)) = \frac{1}{|a|}\Phi\left(\frac{\nu}{a}\right)$$

and

$$\mathcal{F}^{-1}(\Phi(a\nu)) = \frac{1}{|a|}\phi\left(\frac{\nu}{a}\right)$$

The Convolution Theorem

- Convolution in space = multiplication in frequency

$$\mathcal{F}(\phi(x) \otimes h(x)) = \Phi(\nu)H(\nu)$$

- Convolution in frequency = multiplication in space

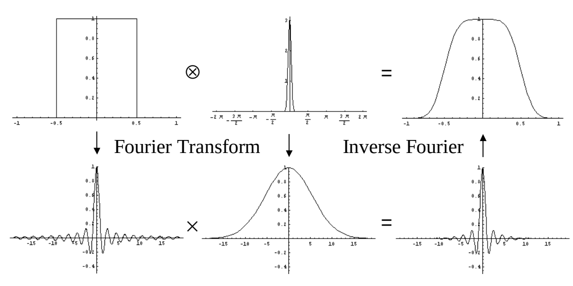
$$\mathcal{F}^{-1}(\Phi(\nu) \otimes H(\nu)) = \phi(x)h(x)$$

Revisiting Convolution

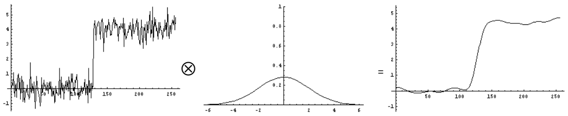
now see convolution as a transform, a multiplication, and an inverse transform.

$$\phi(x)\boxed{\otimes}h(x) = \mathcal{F}^{-1}(\mathcal{F}(\phi(x))\mathcal{F}(h(x)))$$

Revisiting Convolution (2)

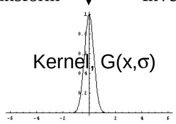


Revisiting Convolution (3)



Input, $\phi(x)$ $\otimes$ Kernel, $g(x,\sigma)$ $= \phi(x) \otimes g(x,\sigma)$

↓ Fourier Transform ↓ Inverse Fourier ↑

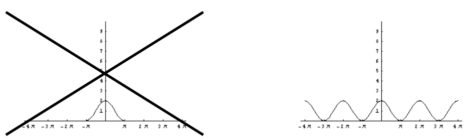


F(Input), $\Phi(x)$ $\times$ Kernel, $G(x,\sigma)$ $= \Phi(x) \times G(\nu,\sigma)$

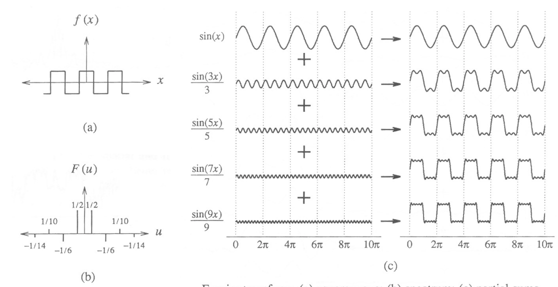
Discrete Fourier Transforms

- Periodic.
  - Assume that the function is a single period of a infinitely repeating function.
  - Or: Think of it as an image that wraps onto itself like a doughnut (torus).
- Discrete.
  - If there are n samples in the spatial domain, there will be n samples in frequency domain, too.

Discrete Fourier Transforms (1)



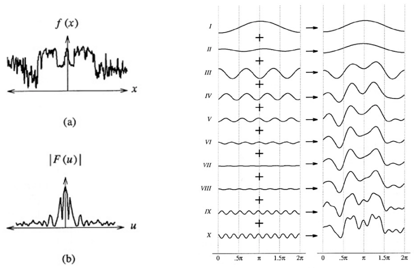
Discrete Fourier Transforms (2)



Fourier transform. (a) square wave; (b) spectrum; (c) partial sums.

(from G. Wolberg, 1990, Digital Image Warping, IEEE Press)

Discrete Fourier Transforms (3)

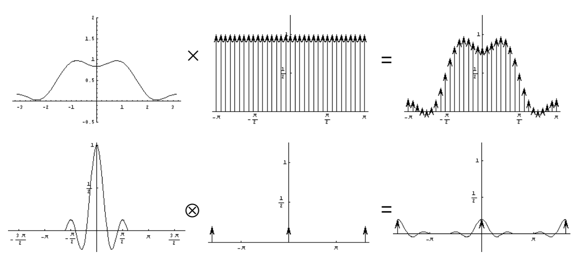


Fourier transform. (a) aperiodic signal; (b) spectrum; (c) partial sums.

(from G. Wolberg, 1990, Digital Image Warping, IEEE Press)

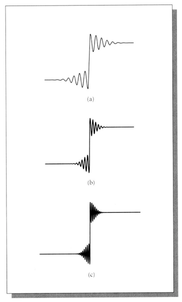
Sampling

now see sampling as a multiplication with frequency issues.



Sampling (2)

Note: adding more (higher) frequencies alters the period of the ringing, but does not reduce the amplitude.



(from A. Glassner, 1995, Principles of Digital Image Synthesis, Morgan Kaufman)

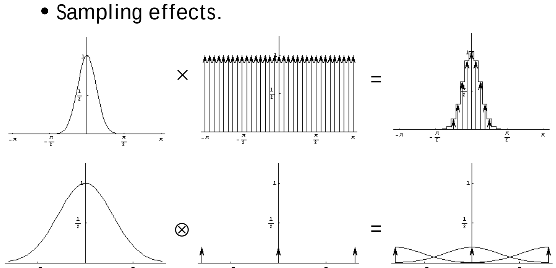
Sampling (3)

- Bandlimited.
  - *A signal ϕ is considered* bandlimited *if its Fourier transform, $F(\phi(x)) = \Phi(\omega)$ satisfies the following condition*

    $\Phi(\omega) = 0$ *and* $\Phi(-\omega) = 0$ *for all* $\omega > \omega_{limit}$
- Satisfies the Nyquist criterion.
  - *The discrete signal does not contain frequencies higher than 1/2 the sampling frequency*

Aliasing

- Signal that isn't Bandlimited.
- Sampling effects.



Sources of Aliasing

Non-bandlimited signal

Low sampling rate (below Nyquist)

Non perfect reconstruction

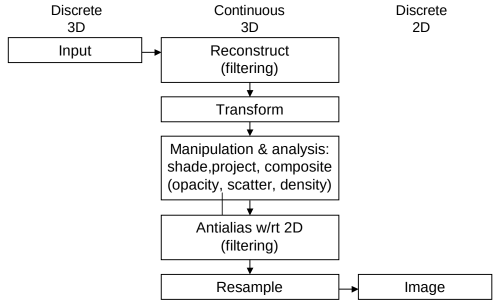
Possible Errors

- Post-aliasing
  - *Reconstruction filter passes frequencies beyond the Nyquist frequency (of duplicated frequency spectrum) => frequency components of the original signal appear in the reconstructed signal at different frequencies.*
- Smoothing
  - *Frequencies below the Nyquist frequency are attenuated.*

Possible Errors(2)

- Ringing (overshoot)
  - *Occurs when trying to sample/reconstruct discontinuity.*
- Anisotropy
  - *Caused by non-spherically symmetric filters.*
  - *Requires filters that are invariant with respect to rotation.*

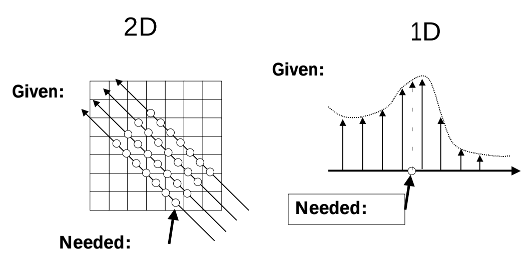
the pipeline (westover '91)



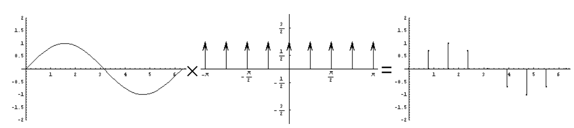
input

- Bandlimited.
- Appropriately sampled: above the Nyquist frequency.

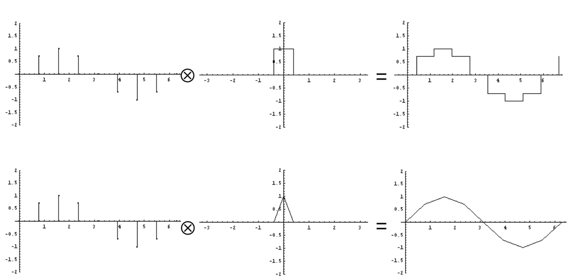
Reconstruction - Interpolation



Example in 1D



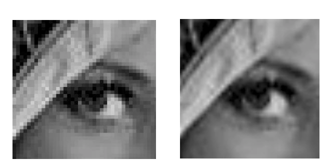
How? Convolution



Interpolation (summary)

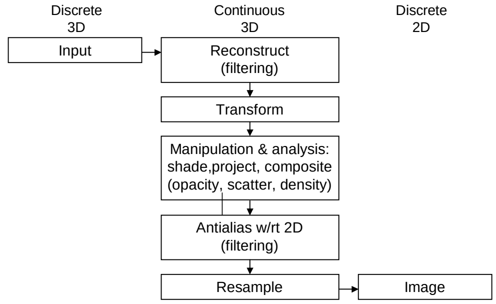
- Very important; regardless of algorithm
  expensive => done very often for one image
- Requirements for good reconstruction
  - performance
  - stability of the numerical algorithm
  - accuracy

Nearest neighbor



Linear

the pipeline (westover '91)



Transformation magnify / minify

- Remember, spatial scaling = inverse frequency scale.
- Magnification / scaling of the reconstructed input ⇒ transformation / minimization of the sampling function.
- Minification / scaling of the reconstructed input ⇒ transformation / magnification of the sampling function.

Resample

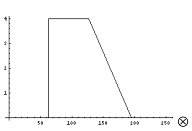
- with antialiasing
- without antialiasing

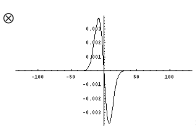
Summary

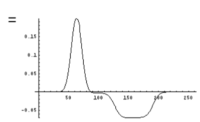
For (n = 1; n < image_size; n++)

gradx[n] = image[n-1] * -0.5 + image[n+1] * 0.5;

You're really approximating…







With all the consequences in the frequency domain!

Summary

- Convolution is a basic operation, used in:
  - *interpolation, reconstruction.*
  - *Noise filtering.*
  - *Differentiation, measurement.*
  - *Statistics.*
- The frequency domain.
  - *It exists.*
  - *Operations in discrete images have frequency based consequences.*
  - *It's happening whether you're watching for it or not.*